Figures

September 1, 2021/updated March 23, 2023

figs

The Art of Linear Algebra
Original illustrations

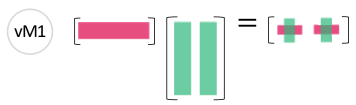
2022/9/22 Matrix World Update

2022/10/6 Separation of pics

2022/11/7 Color change for gray scale

2023/3/4 Adjustment for PREMUS (paper)

illust-p1.eps



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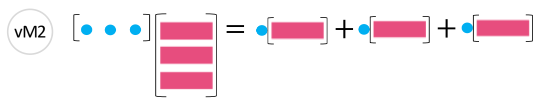

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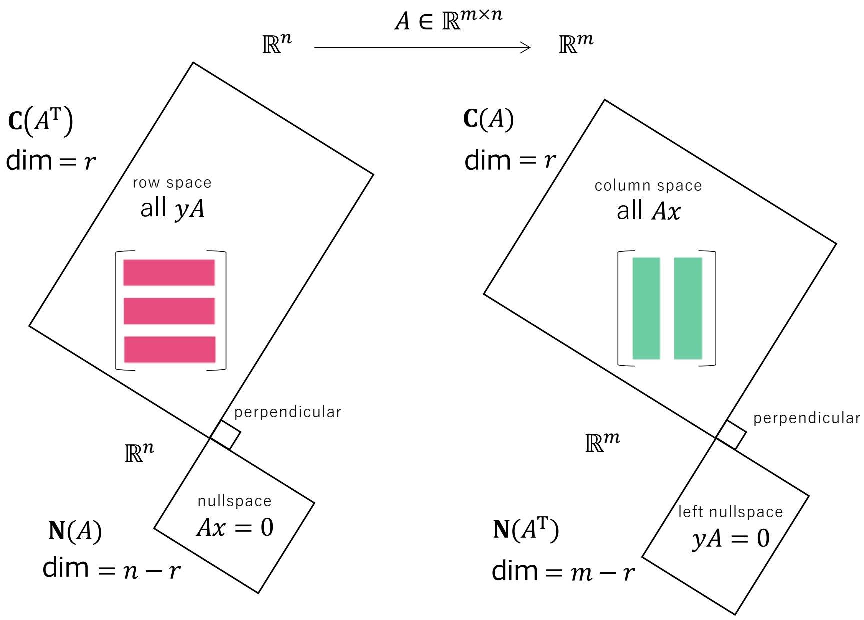

$$\mathbb{R}^n = \mathbf{N}(A) + \mathbf{C}(A^{\mathrm{T}}) \qquad \mathbb{R}^m = \mathbf{C}(A) + \mathbf{N}(A^{\mathrm{T}})$$
$$\mathbf{N}(A) \perp \mathbf{C}(A^{\mathrm{T}}) \qquad \boldsymbol{C}(A) \perp \boldsymbol{N}(A^{\mathrm{T}})$$

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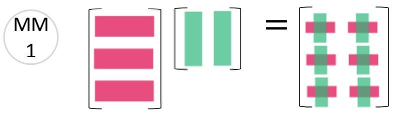
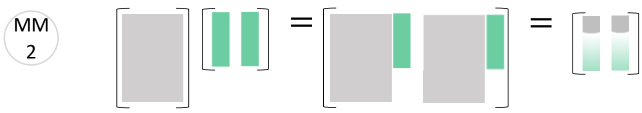


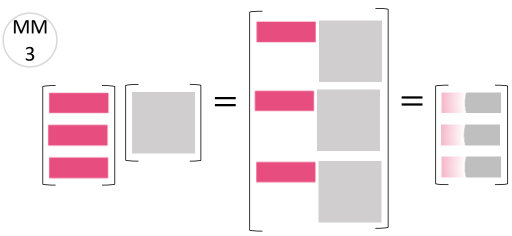
Every element becomes a dot product of row vector and column vector.

$$\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}\begin{bmatrix}x_1 & y_1\\x_2 & y_2\end{bmatrix} = \begin{bmatrix}(x_1{+}2x_2) & (y_1{+}2y_2)\\(3x_1{+}4x_2) & (3y_1{+}4y_2)\\(5x_1{+}6x_2) & (5y_1{+}6y_2)\end{bmatrix}$$



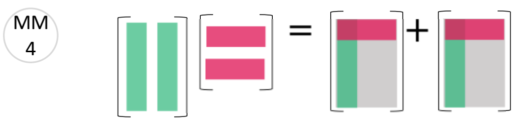
$\boldsymbol{Ax}$ and $\boldsymbol{Ay}$ are linear combinations of columns of A.

$$\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}\begin{bmatrix}x_1 & y_1\\x_2 & y_2\end{bmatrix} = A[\boldsymbol{x} \quad \boldsymbol{y}] = [A\boldsymbol{x} \quad A\boldsymbol{y}]$$



The produced rows are linear combinations of rows.

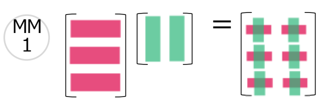
$$\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}\begin{bmatrix}x_1 & y_1\\x_2 & y_2\end{bmatrix} = \begin{bmatrix}\boldsymbol{a}_1^*\\\boldsymbol{a}_2^*\\\boldsymbol{a}_3^*\end{bmatrix}X = \begin{bmatrix}\boldsymbol{a}_1^*X\\\boldsymbol{a}_2^*X\\\boldsymbol{a}_3^*X\end{bmatrix}$$



Multiplication AB is broken down to a sum of rank 1 matrices.

$$\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}\begin{bmatrix}b_{11} & b_{12}\\b_{21} & b_{22}\end{bmatrix} = [\boldsymbol{a}_1 \quad \boldsymbol{a}_2]\begin{bmatrix}\boldsymbol{b}_1^*\\\boldsymbol{b}_2^*\end{bmatrix} = \boldsymbol{a}_1\boldsymbol{b}_1^* + \boldsymbol{a}_2\boldsymbol{b}_2^*$$
$$= \begin{bmatrix}1\\3\\5\end{bmatrix}[b_{11} \quad b_{12}] + \begin{bmatrix}2\\4\\6\end{bmatrix}[b_{21} \quad b_{22}] = \begin{bmatrix}b_{11} & b_{12}\\3b_{11} & 3b_{12}\\5b_{11} & 5b_{12}\end{bmatrix} + \begin{bmatrix}2b_{21} & 2b_{22}\\4b_{21} & 4b_{22}\\6b_{21} & 6b_{22}\end{bmatrix}$$

illust-p13.eps



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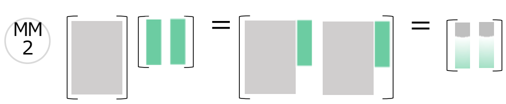

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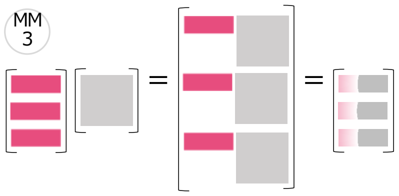

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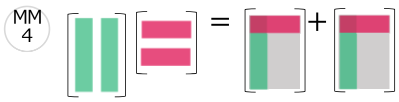

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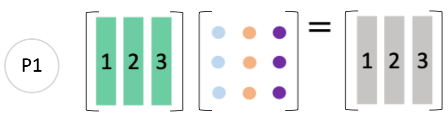
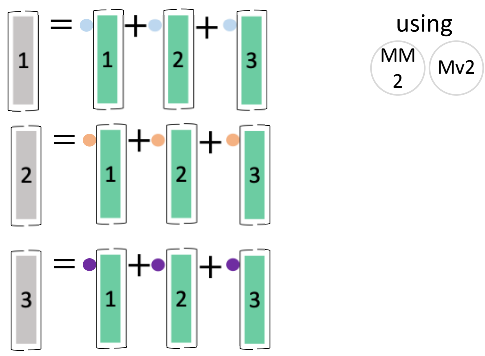
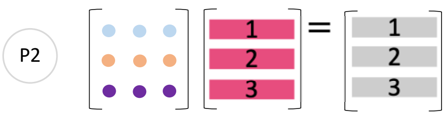
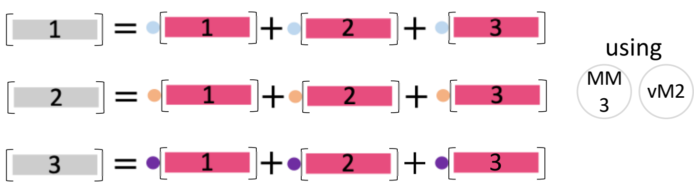


Operations from the right act on the columns of the matrix. This expression can be seen as the three linear combinations in the right in one formula.





Operations from the left act on the rows of the matrix. This expression can be seen as the three linear combinations in the right in one formula.



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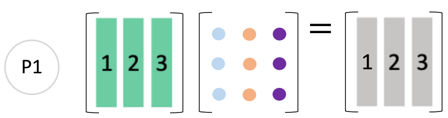

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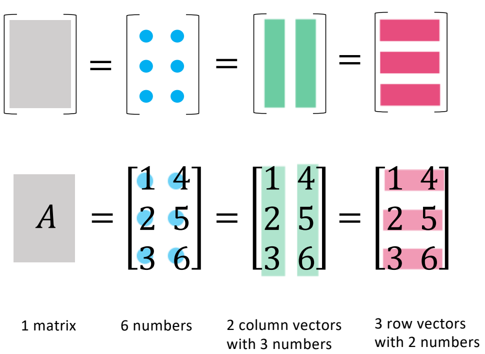
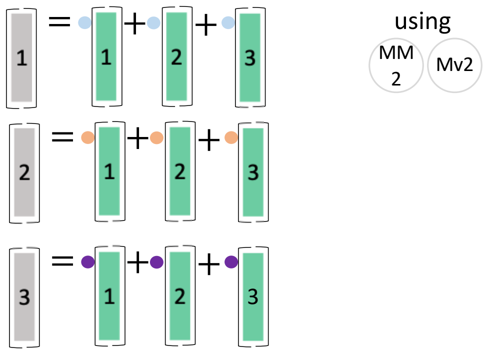


illust-p2.eps



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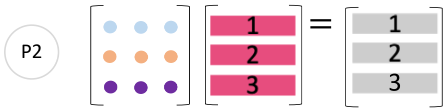

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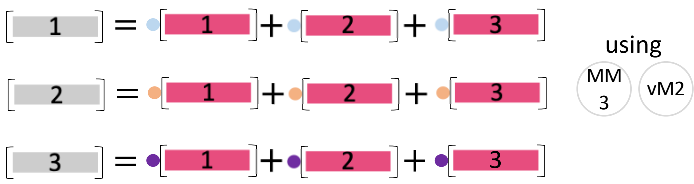

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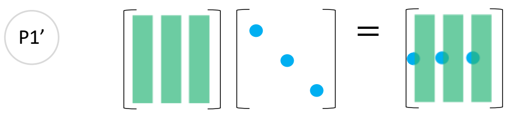
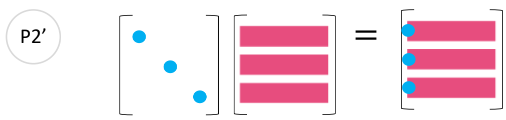


Applying a diagonal matrix from the right scales each column.

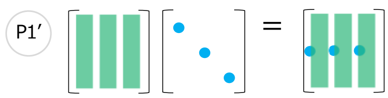
$$AD = [\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \boldsymbol{a}_3]\begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix} = [d_1\boldsymbol{a}_1 \quad d_2\boldsymbol{a}_2 \quad d_3\boldsymbol{a}_3]$$



Applying a diagonal matrix from the left scales each row.

$$DB = \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix}\begin{bmatrix} \boldsymbol{b}_1^* \\ \boldsymbol{b}_2^* \\ \boldsymbol{b}_3^* \end{bmatrix} = \begin{bmatrix} d_1\boldsymbol{b}_1^* \\ d_2\boldsymbol{b}_2^* \\ d_3\boldsymbol{b}_3^* \end{bmatrix}$$

illust-p23.eps



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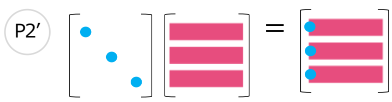
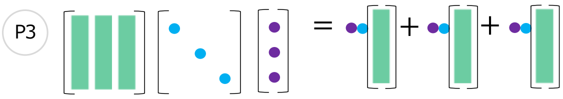


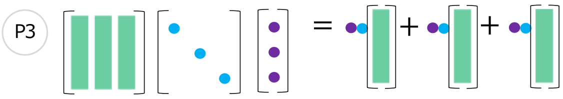
illust-p25.eps



This pattern makes another combination of columns.
You will encounter this in differential/recurrence equations.

$$XD\boldsymbol{c} = [\boldsymbol{x_1} \quad \boldsymbol{x_2} \quad \boldsymbol{x_3}]\begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = c_1 d_1 \boldsymbol{x_1} + c_2 d_2 \boldsymbol{x_2} + c_3 d_3 \boldsymbol{x_3}$$

illust-p26.eps



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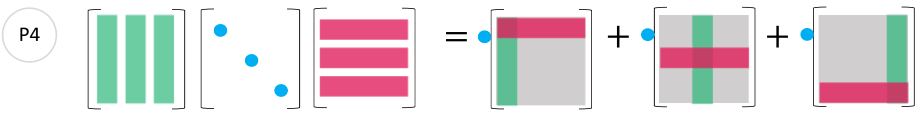
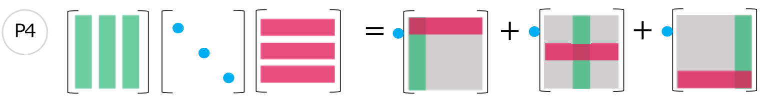


A matrix is broken down to a sum of rank 1 matrices,
as in singular value/eigenvalue decomposition.

$$U\Sigma V^{\mathrm{T}} = [\boldsymbol{u_1} \quad \boldsymbol{u_2} \quad \boldsymbol{u_3}]\begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix}\begin{bmatrix} \boldsymbol{v}_1^{\mathrm{T}} \\ \boldsymbol{v}_2^{\mathrm{T}} \\ \boldsymbol{v}_3^{\mathrm{T}} \end{bmatrix} = \sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^{\mathrm{T}} + \sigma_2 \boldsymbol{u}_2 \boldsymbol{v}_2^{\mathrm{T}} + \sigma_3 \boldsymbol{u}_3 \boldsymbol{v}_3^{\mathrm{T}}$$

illust-p28.eps



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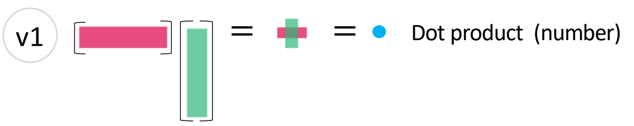
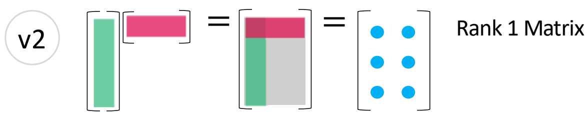




Dot product ($\boldsymbol{a} \cdot \boldsymbol{b}$) is expressed as $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}$ in matrix language and yields a number.

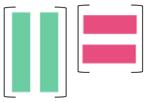
$\boldsymbol{a}\boldsymbol{b}^{\mathrm{T}}$ is a matrix ($\boldsymbol{a}\boldsymbol{b}^{\mathrm{T}} = A$). If neither a, b are 0, the result A is a rank 1 matrix.

$$[1 \quad 2 \quad 3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 + 2x_2 + 3x_3$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}[x \quad y] = \begin{bmatrix} x & y \\ 2x & 2y \\ 3x & 3y \end{bmatrix}$$

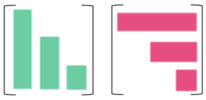
illust-p3.eps

$$A = CR$$



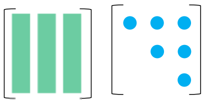
Independent columns in $\boldsymbol{C}$
Row echelon form in $\boldsymbol{R}$
Leads to column rank = row rank

$$A = LU$$



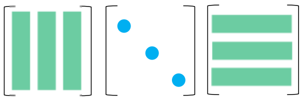
$\boldsymbol{LU}$ decomposition from
Gaussian elimination
(Lower triangular) (Upper triangular)

$$A = QR$$



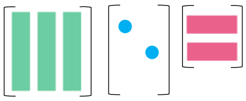
$\boldsymbol{QR}$ decomposition as
Gram-Schmidt orthogonalization
Orthogonal $\boldsymbol{Q}$ and triangular $\boldsymbol{R}$

$$S = Q\Lambda Q^{\mathrm{T}}$$



Eigenvalue decomposition
of a symmetric matrix $\boldsymbol{S}$
Eigenvectors in $\boldsymbol{Q}$, eigenvalues in Λ

$$A = U\Sigma V^{\mathrm{T}}$$



Singular value decomposition
of all matrices $\boldsymbol{A}$
Singular values in Σ

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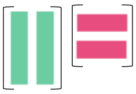

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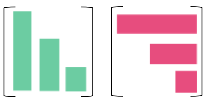

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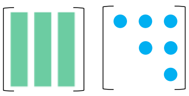

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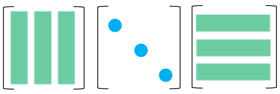

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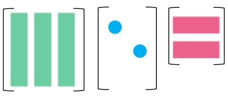

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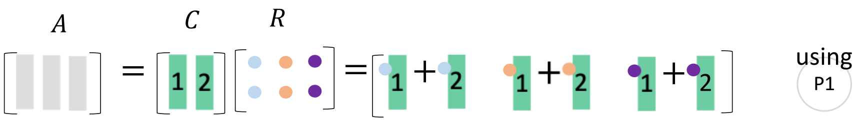

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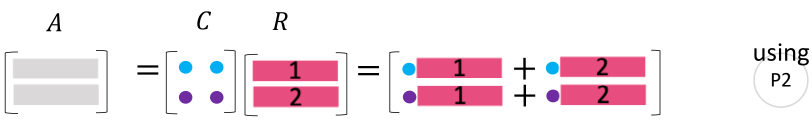

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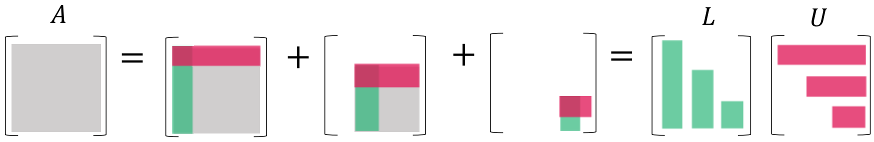

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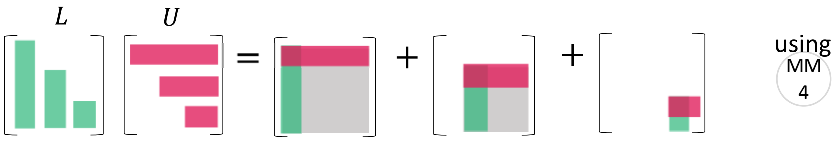

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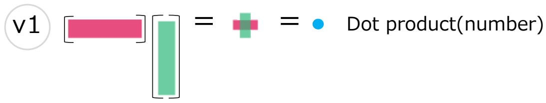

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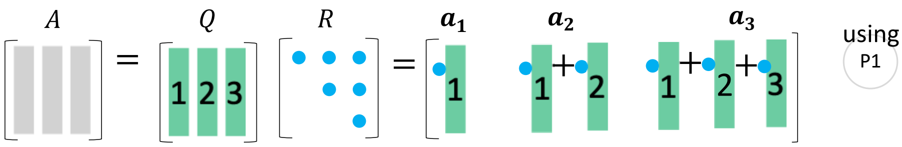

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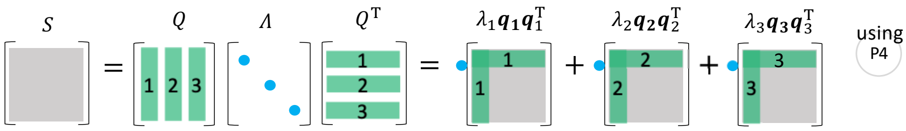

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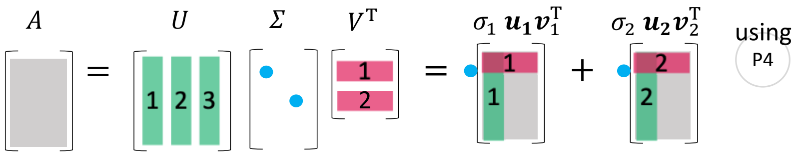

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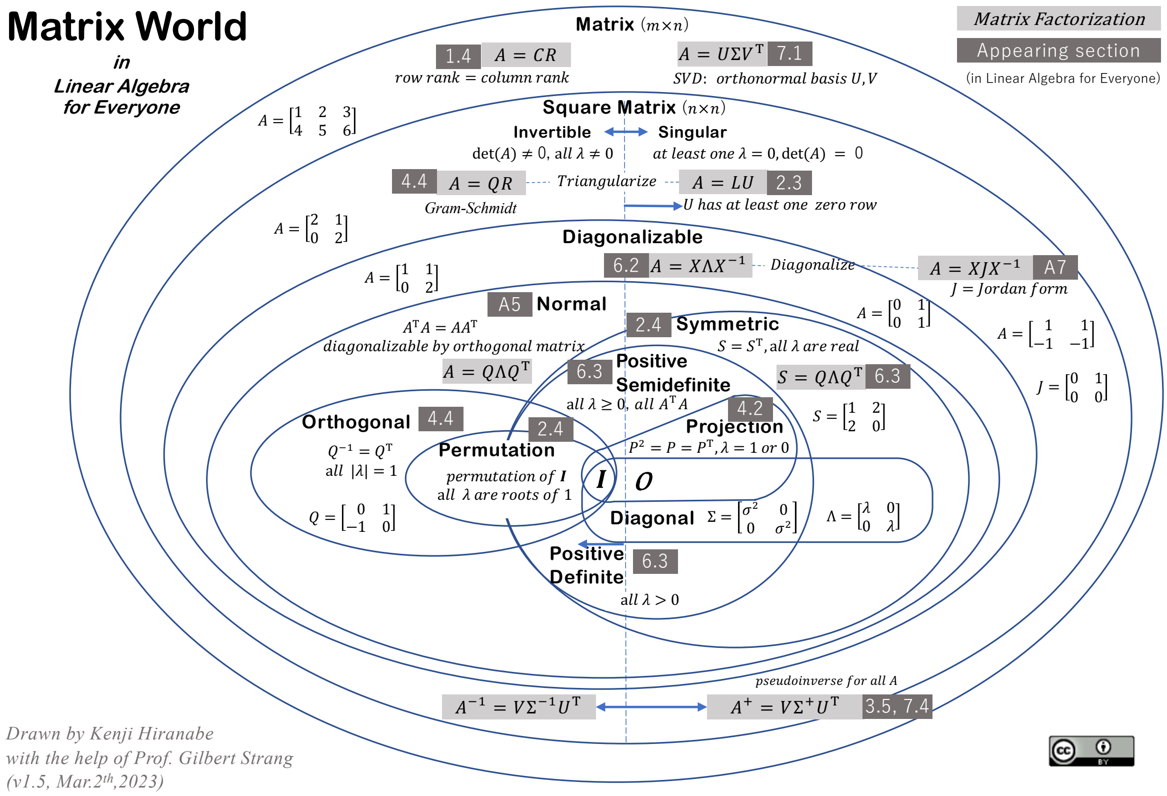

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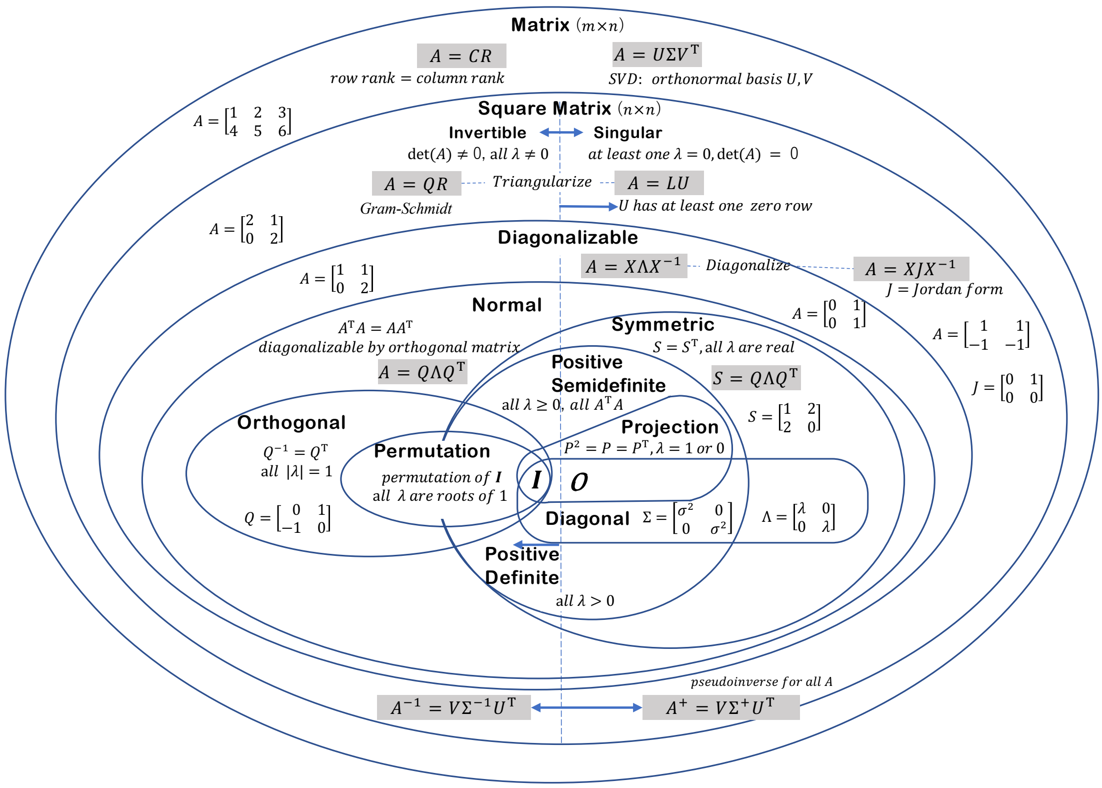

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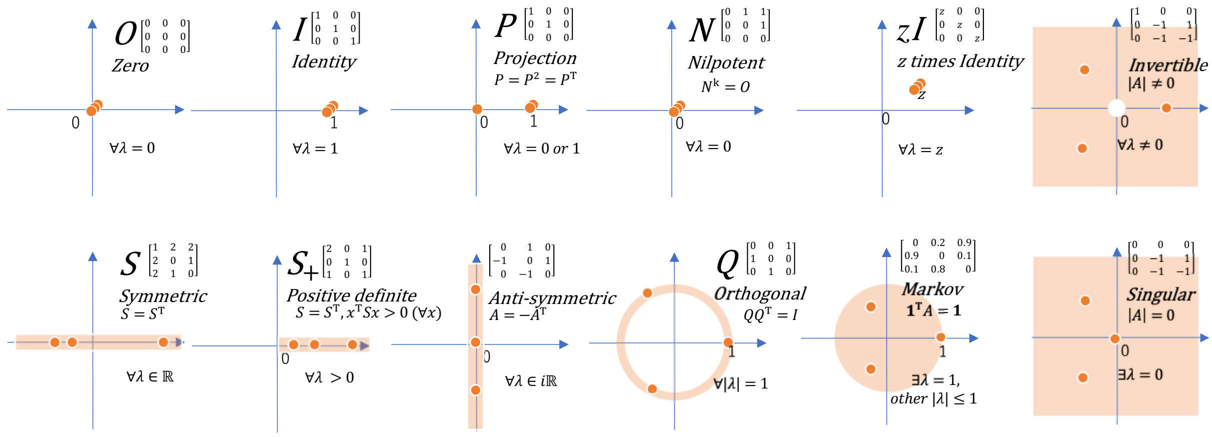

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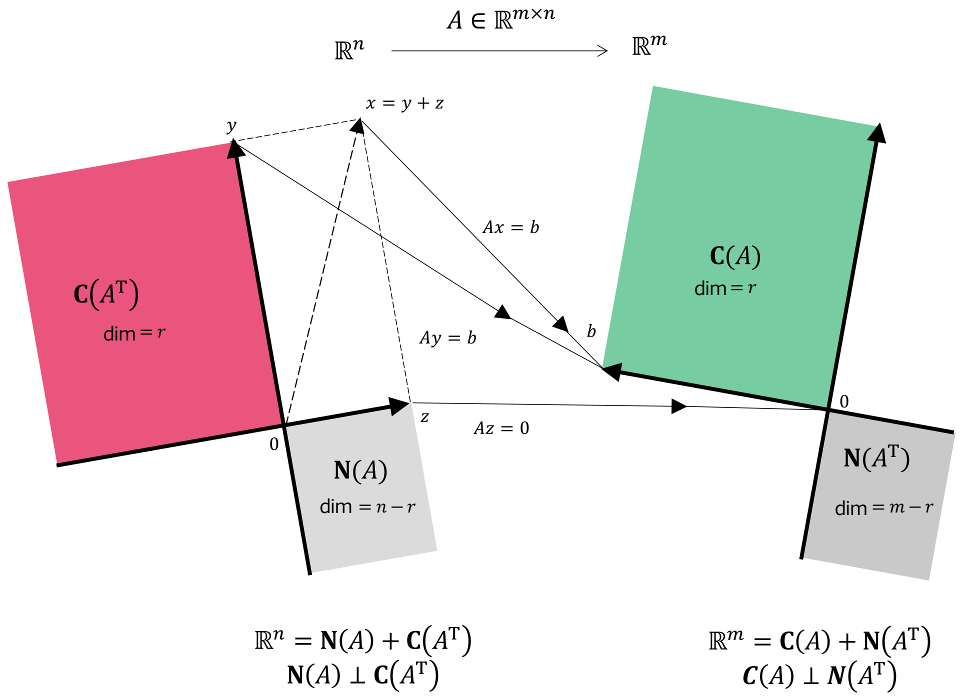

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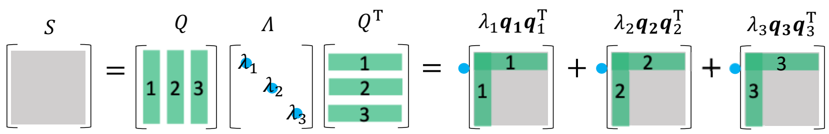

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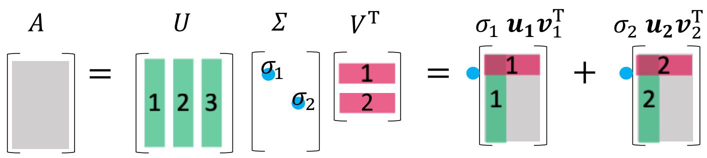

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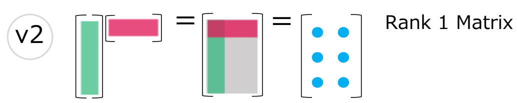

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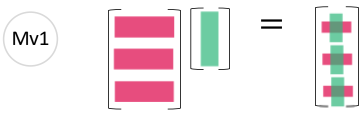
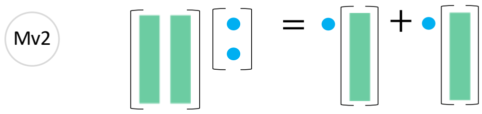




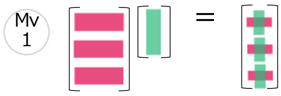
The row vectors of A are multiplied by a vector $\boldsymbol{x}$ and become the three dot-product elements of $A\boldsymbol{x}$.

The product $A\boldsymbol{x}$ is a linear combination of the column vectors of A.

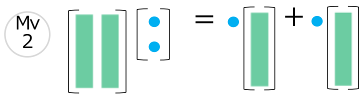
$$A\boldsymbol{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (x_1 + 2x_2) \\ (3x_1 + 4x_2) \\ (5x_1 + 6x_2) \end{bmatrix}$$

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

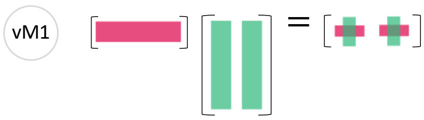
illust-p6.eps



illust-p7.eps

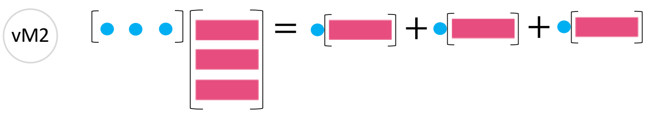


illust-p8.eps



$$\boldsymbol{y}A = [y_1 \quad y_2 \quad y_3] \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = [(y_1 + 3y_2 + 5y_3) \quad (2y_1 + 4y_2 + 6y_3)]$$

A row vector $\boldsymbol{y}$ is multiplied by the two column vectors of A and become the two dot-product elements of $\boldsymbol{y}A$.



$$\boldsymbol{y}A = [y_1 \quad y_2 \quad y_3] \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = y_1[1 \quad 2] + y_2[3 \quad 4] + y_3[5 \quad 6]$$

The product $\boldsymbol{y}A$ is a linear combination of the row vectors of A.

illust-p9.eps